

ON THE HARDER-NARASIMHAN FILTRATION FOR FINITE DIMENSIONAL REPRESENTATIONS OF QUIVERS

ALFONSO ZAMORA

ABSTRACT. We prove that the Harder-Narasimhan filtration for an unstable finite dimensional representation of a finite quiver coincides with the filtration associated to the 1-parameter subgroup of Kempf, which gives maximal instability in the sense of Geometric Invariant Theory for the corresponding point in the parameter space where these objects are parametrized in the construction of the moduli space.

INTRODUCTION

Let Q be a finite quiver, given by a finite set of vertices and arrows between them, and a representation of Q on finite dimensional k -vector spaces, where k is an algebraically closed field of arbitrary characteristic. There exists a notion of stability for such representations given by King ([Ki]) and, more generally by Reineke ([Re]) (both particular cases of the abstract notion of stability for an abelian category that we can find in [Ru]), and a notion of the existence of a unique Harder-Narasimhan filtration with respect to that stability condition.

We consider the construction of a moduli space for these objects by King ([Ki]) and associate to an unstable representation an unstable point, in the sense of Geometric Invariant Theory, in a parameter space where a group acts. Then, the 1-parameter subgroup given by Kempf ([Ke]), which is maximally destabilizing in the GIT sense, gives a filtration of subrepresentations and we prove that it coincides with the Harder-Narasimhan filtration for that representation.

This article makes use of the same techniques that a previous work of the author in collaboration with T. Gómez and I. Sols ([GSZ]). In that article, we considered an unstable torsion free sheaf E over a smooth projective variety X . There, we proved that the filtration associated to the 1-parameter subgroup given by Kempf, coincides with the Harder-Narasimhan filtration of E with the definition of stability given by Gieseker.

The definition of stability for a representation of a quiver (c.f. Definition 1.1) contains two sets of parameters, the coefficients of the linear functions Θ and σ . In

[Ke], the 1-parameter subgroup is taken to maximize certain function which depends on the choice of a linearization of the action of the group we are taking the quotient by, and a "length" in the set of 1-parameter subgroups (c.f. Definition 3.1). In the case of sheaves the group is $SL(N)$, which is simple, so any such length is unique up to multiplication by a scalar, whereas for finite dimensional representations of quivers we quotient by a product of general linear groups, so we have to choose a scalar for each factor in the choice of a length. Hence, we put the positive coefficients of σ precisely as these scalars and consider a particular linearization depending on σ and Θ , in order to relate the Harder-Narasimhan filtration of a representation with the filtration given by [Ke] (c.f. Theorem 5.3).

Acknowledgments. The author wishes to thank L. Álvarez-Cónsul and T. Gómez for useful discussions. This work has been supported by project MTM2010-17389 and ICMAT Severo Ochoa project SEV-2011-0087 granted by Ministerio de Economía y Competitividad. The author was also supported by a FPU grant from the Spanish Ministerio de Educación.

1. HARDER-NARASIMHAN FILTRATION FOR REPRESENTATIONS OF QUIVERS

A finite quiver Q is given by a finite set of vertices Q_0 and a finite set of arrows Q_1 . The arrows will be denoted by $(\alpha : v_i \rightarrow v_j) \in Q_1$. We denote by $\mathbb{Z}Q_0$ the free abelian group generated by Q_0 .

Fix k , an algebraically closed field of arbitrary characteristic. Let $\text{mod } kQ$ be the category of finite-dimensional representations of Q over k . Such category is an abelian category and its objects are given by tuples

$$M = ((M_v)_{v \in Q_0}, (M_\alpha : M_{v_i} \rightarrow M_{v_j})_{\alpha : v_i \rightarrow v_j})$$

of finite dimensional k -vector spaces and k -linear maps between them. The dimension vector of a representation is given by $\underline{\dim} M = \sum_{v \in Q_0} \dim_k M_v \cdot v \in \mathbb{N}Q_0$.

Let Θ be a set of numbers Θ_v for each $v \in Q_0$ and define a linear function $\Theta : \mathbb{Z}Q_0 \rightarrow \mathbb{Z}$, by

$$\Theta(M) := \Theta(\underline{\dim} M) = \sum_{v \in Q_0} \Theta_v \dim_k M_v .$$

Let σ be a set of strictly positive numbers σ_v for each $v \in Q_0$, and define a (strictly positive) linear function $\sigma : \mathbb{Z}Q_0 \rightarrow \mathbb{Z}$, by

$$\sigma(M) := \sigma(\underline{\dim} M) = \sum_{v \in Q_0} \sigma_v \dim_k M_v .$$

We call $\sigma(M)$ *the total dimension of M* . we will refer to Θ and σ indistinctly meaning the sets of numbers or the linear functions.

For a non-zero representation M of Q over k , define its slope by

$$\mu_{(\Theta, \sigma)}(M) := \frac{\Theta(M)}{\sigma(M)} .$$

Definition 1.1. *A representation M of Q over k is (Θ, σ) -semistable if for all non-zero subrepresentations M' of M , we have*

$$\mu_{(\Theta, \sigma)}(M') \leq \mu_{(\Theta, \sigma)}(M) .$$

If the inequality is strict for every non-zero subrepresentation, we say that M is (Θ, σ) -stable.

Lemma 1.2. *If we multiply the linear function Θ by a non-negative integer, or if we add an integer multiple of the strictly positive linear function σ to Θ , the semistable (resp. stable) representations remain semistable (resp. stable).*

Proof. Let $\Theta' = a \cdot \Theta + b \cdot \sigma$, $a, b \in \mathbb{Z}$, $a > 0$ be another linear function and note that

$$\begin{aligned} \frac{\Theta'(M')}{\sigma(M')} \leq \frac{\Theta'(M)}{\sigma(M)} &\Leftrightarrow \frac{a \cdot \Theta(M') + b \cdot \sigma(M)}{\sigma(M')} \leq \frac{a \cdot \Theta(M) + b \cdot \sigma(M)}{\sigma(M)} \\ &\Leftrightarrow \frac{\Theta(M')}{\sigma(M')} \leq \frac{\Theta(M)}{\sigma(M)} . \end{aligned}$$

■

Remark 1.3. *The definition of stability which appears in [Ki] and [Re] considers $\sigma_v = 1$ for each $v \in Q_0$, although we consider a strictly positive linear function σ in general. The notation of σ agrees with [AC], [ACGP], [Sch], while Θ agrees with [Ki] and [Re] but in the other references it is substituted by different notations closer to classical moduli problems where the stability notion depends on parameters (τ -stability or ρ -stability).*

Lemma 1.4. [Ru, Definition 1], [Re, Lemma 4.1] *Let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be a short exact sequence of non-zero representations of Q over k . Then $\mu_{(\Theta, \sigma)}(X) < \mu_{(\Theta, \sigma)}(Y)$ if and only if $\mu_{(\Theta, \sigma)}(X) < \mu_{(\Theta, \sigma)}(Z)$ if and only if $\mu_{(\Theta, \sigma)}(Y) < \mu_{(\Theta, \sigma)}(Z)$.*

Proof. Note that $\sigma(Y) = \sigma(X) + \sigma(Z)$ and, therefore

$$\mu_{(\Theta, \sigma)}(Y) = \frac{\Theta(Y)}{\sigma(Y)} = \frac{\Theta(X) + \Theta(Z)}{\sigma(X) + \sigma(Z)} ,$$

from which the statement follows. ■

Theorem 1.5. [Ru, Theorem 2], [Re, Lemma 4.7] *Given linear functions Θ and σ , (being σ strictly positive), every representation M of Q over k has a unique filtration*

$$0 \subset M_1 \subset M_2 \subset \dots \subset M_t \subset M_{t+1} = M$$

verifying the following properties, where $M^i := M_i/M_{i-1}$

- (1) $\mu_{(\Theta, \sigma)}(M^1) > \mu_{(\Theta, \sigma)}(M^2) > \dots > \mu_{(\Theta, \sigma)}(M^t) > \mu_{(\Theta, \sigma)}(M^{t+1})$
- (2) *The quotients M^i are (Θ, σ) -semistable*

This filtration is called the Harder-Narasimhan filtration of M (with respect to Θ and σ).

Proof. Using Lemma 1.4 we can prove the existence of a unique subrepresentation M_1 , whose slope is maximal among all the subrepresentations of M , and of maximal total dimension $\sigma(M_1)$ between those of maximal slope (c.f. [Ru, Proposition 1.9], [Re, Lemma 4.4]). Then, proceed by recursion on the quotient M/M_1 . ■

2. MODULI SPACE OF REPRESENTATIONS OF QUIVERS

Fix k an algebraically closed field of arbitrary characteristic. Fix a dimension vector $d \in \mathbb{Z}Q_0$ and fix k -vector spaces M_v of dimension d_v for all $v \in Q_0$. Fix linear functions $\Theta, \sigma : \mathbb{Z}Q_0 \rightarrow \mathbb{Z}$, being σ strictly positive. We recall the construction by King (c.f. [Ki]) of a moduli space for representations of Q over k with dimension vector d .

Consider the affine k -space

$$\mathcal{R}_d(Q) = \bigoplus_{\alpha: v_i \rightarrow v_j} \text{Hom}_k(M_{v_i}, M_{v_j})$$

whose points parametrize representations of Q on the k -vector spaces M_v . The reductive linear algebraic group

$$G_d = \prod_{v \in Q_0} GL(M_v)$$

acts on $\mathcal{R}_d(Q)$ by

$$(g_{v_i})_{v_i} \cdot (M_\alpha)_\alpha = (g_{v_j} M_\alpha g_{v_i}^{-1})_{\alpha: v_i \rightarrow v_j}$$

and the G_d -orbits of M in $\mathcal{R}_d(Q)$ correspond bijectively to the isomorphism classes $[M]$ of k -representations of Q with dimension vector d . We will use Geometric Invariant Theory to take the quotient of $\mathcal{R}_d(Q)$ by G_d and construct a moduli space of representations of the quiver Q on the k -vector spaces M_v .

The action of G_d on the affine space $\mathcal{R}_d(Q)$ can be lifted by a character χ to the (necessarily trivial) line bundle L required by the Geometric Invariant Theory. Note that the subgroup of the diagonal scalar matrices in G_d ,

$$\Delta = \{(t1, \dots, t1) : t \in k^*\},$$

acts trivially on $\mathcal{R}_d(Q)$. Then, we have to choose χ in such a way that Δ acts trivially on the fiber, in other words, $\chi(\Delta) = 1$.

Then, using the linear functions Θ and σ , consider the character

$$\chi_{(\Theta, \sigma)}((g_v)_v) := \prod_{v \in Q_0} \det(g_v)^{(\Theta(d)\sigma_v - \sigma(d)\Theta_v)}$$

of G_d , and note that $\chi_{(\Theta, \sigma)}(\Delta) = 1$, because $\sum_{v \in Q_0} (\Theta(d)\sigma_v - \sigma(d)\Theta_v) \cdot d_v = 0$.

Definition 2.1. [Ki, Definition 2.1] *A point $x \in \mathcal{R}_d(Q)$ is χ -semistable if there is a relative invariant $f \in k[\mathcal{R}_d(Q)]^{G_d, \chi_{(\Theta, \sigma)}^n}$ with $n \geq 1$, such that $f(x) \neq 0$.*

The algebraic quotient will be given by

$$\mathcal{R}_d(Q) // (G_d, \chi_{\Theta, \sigma}) = \text{Proj} \left(\bigoplus_{n \geq 0} k[\mathcal{R}_d(Q)]^{G_d, \chi_{(\Theta, \sigma)}^n} \right).$$

Proposition 2.2. *A point $x_M \in \mathcal{R}_d(Q)$ corresponding to a representation $M \in \text{mod } kQ$ is $\chi_{(\Theta, \sigma)}$ -semistable (resp. $\chi_{(\Theta, \sigma)}$ -stable) for the action of G_d if and only if M is (Θ, σ) -semistable (resp. (Θ, σ) -stable).*

Proof. It follows easily from [Ki, Proposition 3.1]. In [Ki], given a linear function Θ , a representation M is Θ -semistable if $\Theta(M) = 0$ and for every subrepresentation $M' \subset M$, we have $\Theta(M') \geq 0$ (c.f. [Ki, Definition 1.1]). Then, [Ki, Proposition 3.1] relates the Θ -stability with the χ_Θ -stability, where the character is

$$\chi_\Theta((g_v)_v) := \prod_{v \in Q_0} \det(g_v)^{\Theta_v}.$$

Hence, the $\chi_{(\Theta, \sigma)}$ -stability with the character

$$\chi_{(\Theta, \sigma)}((g_v)_v) := \prod_{v \in Q_0} \det(g_v)^{(\Theta(d)\sigma_v - \sigma(d)\Theta_v)}$$

is equivalent to the (Θ, σ) -stability in Definition 1.1, because for a subrepresentation $M' \subset M$, the expression

$$\sum_{v \in Q_0} (\Theta(M)\sigma_v - \sigma(M)\Theta_v) \cdot \dim M'_v = \Theta(M)\sigma(M') - \sigma(M)\Theta(M') \geq 0$$

is equivalent to

$$\frac{\Theta(M')}{\sigma(M')} \leq \frac{\Theta(M)}{\sigma(M)}.$$

■

Now denote by $\mathcal{R}_d^{(\Theta, \sigma)-ss}(Q)$ the set of $\chi_{(\Theta, \sigma)}$ -semistable points.

Theorem 2.3. [Ki, Proposition 4.3], [Re, Corollary 3.7] *The moduli space $\mathfrak{M}_d^{(\Theta, \sigma)}(Q) = \mathcal{R}_d^{(\Theta, \sigma)-ss}(Q) // G_d$ is a projective variety which parametrizes S -equivalence classes of (Θ, σ) -semistable representations of Q of dimension vector d .*

By the Hilbert-Mumford criterion we can characterize $\chi_{(\Theta, \sigma)}$ -semistable points by its behavior under the action of 1-parameter subgroups. A 1-parameter subgroup of $G_d = \prod_{v \in Q_0} GL(M_v)$ is a non-trivial homomorphism $\Gamma : k^* \rightarrow G_d$. There exist bases of the vector spaces M_v such that Γ takes the diagonal form

$$\begin{pmatrix} t^{\Gamma_{v_1, 1}} & & \\ & \ddots & \\ & & t^{\Gamma_{v_1, t_1+1}} \end{pmatrix} \times \cdots \times \begin{pmatrix} t^{\Gamma_{v_s, 1}} & & \\ & \ddots & \\ & & t^{\Gamma_{v_s, t_s+1}} \end{pmatrix}$$

where $v_1, \dots, v_s \in Q_0$ are the vertices of the quiver.

Let $x \in \mathcal{R}_d(Q)$ and suppose that $\lim_{t \rightarrow 0} \Gamma \cdot x$ exists and is equal to x_0 . Then x_0 is a fixed point for the action of Γ , and Γ acts on the fiber of the trivial line bundle over x_0 as multiplication by t^a . Define the following numerical function,

$$\mu_{\chi_{(\Theta, \sigma)}}(x, \Gamma) = -a.$$

The next proposition establishes the so-called "Hilbert-Mumford numerical criterion":

Proposition 2.4. [Ki, Proposition 2.5] *A point $x_M \in \mathcal{R}_d(Q)$ corresponding to a representation M is $\chi_{(\Theta, \sigma)}$ -semistable if and only if every 1-parameter subgroup Γ of G_d , for which $\lim_{t \rightarrow 0} \Gamma(t) \cdot x_M$ exists, satisfies $\mu_{\chi_{(\Theta, \sigma)}}(x_M, \Gamma) \leq 0$.*

Remark 2.5. *Note that in Proposition 2.4 we change the sign of the numerical function $\mu_{\chi_{(\Theta, \sigma)}}(x_M, \Gamma)$ with respect to [Ki] (as we did when changing the character in the proof of Proposition 2.2), in congruence with [Ke] and [GSZ].*

The action of a 1-parameter subgroup Γ of G_d provides a decomposition of each vector space M_v associated to each vertex $v \in Q_0$, in weight spaces

$$M_v = \bigoplus_{n \in \mathbb{Z}} M_v^n$$

where $\Gamma(t)$ acts on the weight space M_v^n as multiplication by t^n . Every 1-parameter subgroup, for which $\lim_{t \rightarrow 0} \Gamma(t) \cdot x$ exists, determines a weighted filtration $M_\bullet \subset M$ of subrepresentations (c.f. [Ki])

$$0 \subset M_1 \subset M_2 \subset \dots \subset M_t \subset M_{t+1} = M$$

where M_i is the subrepresentation with vector spaces $M_{v,i} := \bigoplus_{n \leq i} M_v^n$ for each vertex $v \in Q_0$, and the weight corresponding to each quotient $M^i = M_i/M_{i-1}$ is Γ_i . Note that two 1-parameter subgroups giving the same filtration are conjugated by an element of the parabolic subgroup of G_d defined by the filtration. Therefore, the numerical function $\mu_{\chi_{(\Theta, \sigma)}}(x_M, \Gamma)$, has a simple expression in terms of the filtration $M_\bullet \subset M$ (c.f. calculation in [Ki]):

$$(2.1) \quad \mu_{\chi_{(\Theta, \sigma)}}(x_M, \Gamma) = \sum_{v \in Q_0} [(\Theta(M)\sigma_v - \sigma(M)\Theta_v) \cdot \sum_{i=1}^{t_v+1} \Gamma_{v,i} \dim M_v^i] .$$

Let d_i, d^i be the dimension vectors of the subrepresentation M_i and the quotient $M^i = M_i/M_{i-1}$, respectively. The action of Γ on the point corresponding to a representation M has different weights for each vertex $v \in Q_0$, but collect all different weights Γ_i corresponding to any vertex and form the vector

$$\Gamma = (\Gamma_1, \Gamma_2, \dots, \Gamma_t, \Gamma_{t+1})$$

verifying $\Gamma_1 < \Gamma_2 < \dots < \Gamma_t < \Gamma_{t+1}$. Hence, (2.1) turns out to be

$$(2.2) \quad \mu_{\chi_{(\Theta, \sigma)}}(x_M, \Gamma) = \sum_{i=1}^{t+1} \Gamma_i \cdot [\Theta(M) \cdot \sigma(M^i) - \sigma(M) \cdot \Theta(M^i)] ,$$

and Proposition 2.4 can be rewritten in terms of filtrations of M .

Proposition 2.6. *A point $x_M \in \mathcal{R}_d(Q)$ corresponding to a representation M of Q over k , is $\chi_{(\Theta, \sigma)}$ -semistable if and only if every 1-parameter subgroup Γ of G_d , defining a filtration of subrepresentations of M ,*

$$0 \subset M_1 \subset M_2 \subset \dots \subset M_t \subset M_{t+1} = M ,$$

satisfies that

$$\mu_{\chi_{(\Theta, \sigma)}}(x_M, \Gamma) = \sum_{i=1}^{t+1} \Gamma_i \cdot [\Theta(M) \cdot \sigma(M^i) - \sigma(M) \cdot \Theta(M^i)] \leq 0 .$$

3. KEMPF THEOREM

Given a weighted filtration of M ,

$$0 \subset M_1 \subset M_2 \subset \dots \subset M_t \subset M_{t+1} = M$$

and $\Gamma_1 < \Gamma_2 < \dots < \Gamma_t < \Gamma_{t+1}$, define the following function which we call the *Kempf function*,

$$(3.1) \quad K(x_M, \Gamma) = \frac{\sum_{i=1}^{t+1} \Gamma_i \cdot [\Theta(M) \cdot \sigma(M^i) - \sigma(M) \cdot \Theta(M^i)]}{\sqrt{\sum_{i=1}^{t+1} \sigma(M^i) \cdot \Gamma_i^2}}$$

We recall a theorem by Kempf (c.f. [Ke, Theorem 2.2]) stating that whenever there exists any Γ giving a positive value for the numerator of the Kempf function, there exists a unique parabolic subgroup containing a unique 1-parameter subgroup in each maximal torus, giving maximum in the Kempf function i.e., there exists a unique weighted filtration of M for which the Kempf function achieves its maximum.

The Kempf function (3.1) which appears in [Ke, Theorem 2.2] is a rational function whose numerator is equal to the numerical function $\mu_{\chi(\Theta, \sigma)}(x_M, \Gamma)$ and the denominator is the "length" of the 1-parameter subgroup Γ . Given a reductive algebraic linear group G , there is a notion of "length" defined by Kempf (c.f. [Ke, pg. 305]) in $\Gamma(G)$, the set of all 1-parameter subgroups.

Definition 3.1. *A length is a non-negative function $\|\cdot\|$ on $\Gamma(G)$ with values on the real numbers, invariant by conjugation by rational points of G , and such that for any maximal torus $T \subset G$, there is a positive definite integral valued form (\cdot, \cdot) in $\Gamma(T)$ with $(\Gamma, \Gamma) = \|\Gamma\|^2$, for any $\Gamma \in \Gamma(T)$.*

If G is semisimple in characteristic zero all choices of length will be multiples of the Killing form in the Lie algebra \mathfrak{g} (note that in this case $\Gamma(G) \subset \mathfrak{g}$) and, in general, for an almost simple group in arbitrary characteristic, all lengths differ also by a scalar.

Hence, observe that in the Kempf function (3.1), the denominator of the expression is a length verifying the properties in Definition 3.1. Therefore, we can rewrite [Ke, Theorem 2.2] in our case as follows:

Theorem 3.2. *Given a $\chi_{(\Theta, \sigma)}$ -unstable point $x_M \in \mathcal{R}_d(Q)$ corresponding to a representation M , there exists a unique weighted filtration, i.e. $0 \subset M_1 \subset \dots \subset M_{t+1} = M$ and real numbers $\Gamma_1 < \Gamma_2 < \dots < \Gamma_t < \Gamma_{t+1}$, called the Kempf filtration of M , such that the Kempf function K achieves the maximum among all filtrations and weights verifying $\Gamma_1 < \Gamma_2 < \dots < \Gamma_t < \Gamma_{t+1}$.*

Note that the length we are considering depends on the choice of σ and the Kempf function depends both on the length and the linearization of the group action, hence depends both on Θ and σ . In order to relate the Kempf filtration of M with the Harder-Narasimhan filtration, which also depends on Θ and σ , we dispose the parameters conveniently.

4. RESULTS ON CONVEXITY

Next, we prove a result about convexity for functions which are similar to the Kempf function. The vector which maximizes such functions verifies some properties that will be strongly related to the properties of the Harder-Narasimhan filtration. In this section we recall the results of [GSZ, Section 2].

Consider \mathbb{R}^{t+1} together with an inner product (\cdot, \cdot) defined by the diagonal matrix

$$\begin{pmatrix} b^1 & & 0 \\ & \ddots & \\ 0 & & b^{t+1} \end{pmatrix}$$

where b^i are positive integers. Let

$$\mathcal{C} = \{x \in \mathbb{R}^{t+1} : x_1 < x_2 < \cdots < x_{t+1}\},$$

and $v = (v_1, \dots, v_{t+1}) \in \mathbb{R}^{t+1}$ verifying $\sum_{i=1}^{t+1} b^i v_i = 0$. Define the function

$$\begin{aligned} \mu_v : \overline{\mathcal{C}} &\rightarrow \mathbb{R} \\ \Gamma &\mapsto \mu_v(\Gamma) = \frac{(\Gamma, v)}{\|\Gamma\|} \end{aligned}$$

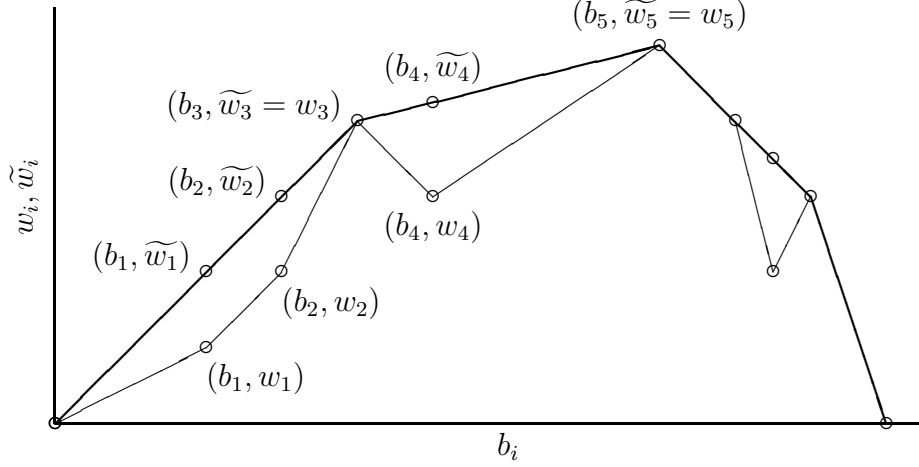
and note that $\mu_v(\Gamma) = \|v\| \cdot \cos(\Gamma, v)$.

We assume that there exists $\Gamma \in \overline{\mathcal{C}}$ with $\mu_v(\Gamma) > 0$ and then, we would like to find a vector $\Gamma \in \overline{\mathcal{C}}$ which maximizes the function μ_v . Define $w^i = -b^i \cdot v_i$, $w_i = w^1 + \cdots + w^i$, $b_i = b^1 + \cdots + b^i$ and draw a graph joining the points with coordinates (b_i, w_i) , each segment having slope $-v_i$. Now draw the convex envelope of this graph (thick line in Figure 1), denoting its coordinates by (b_i, \widetilde{w}_i) , and define

$$\Gamma_i = -\frac{\widetilde{w}_i - \widetilde{w}_{i-1}}{b^i}.$$

In other words, the vector $\Gamma_v = (\Gamma_1, \dots, \Gamma_{t+1})$ defined in this way, verifies that the quantities $-\Gamma_i$ are the slopes of the convex envelope graph defined by v .

Theorem 4.1. [GSZ, Theorem 2.2] *The vector Γ_v defined in this way gives the maximum for the function μ_v on its domain.*

FIGURE 1. Convex envelope Γ_v of the graph defined by v

5. KEMPF FILTRATION IS HARDER-NARASIMHAN FILTRATION

Finally, we study the geometrical properties of the Kempf filtration by associating to it a graph which encodes the two properties satisfied by the Harder-Narasimhan filtration.

Let $\Theta : \mathbb{Z}Q_0 \rightarrow \mathbb{Z}$ be a linear function and let $\sigma : \mathbb{Z}Q_0 \rightarrow \mathbb{Z}$ be a strictly positive linear function. Let M be a representation of Q over an algebraically closed field k of arbitrary characteristic, which is (Θ, σ) -unstable. Consider the $\chi_{(\Theta, \sigma)}$ -unstable point $x_M \in \mathcal{R}_d(Q)$ associated to M , by Proposition 2.2. Let $0 \subset M_1 \subset \dots \subset M_{t+1} = M$ and $\Gamma_1 < \Gamma_2 < \dots < \Gamma_t < \Gamma_{t+1}$ be the Kempf filtration of M , by Theorem 3.2.

Let $M^i = M_i/M_{i-1}$ be the quotients of the filtration. Consider the inner product in \mathbb{R}^{t+1} given by the matrix

$$\begin{pmatrix} \sigma(M^1) & & 0 \\ & \ddots & \\ 0 & & \sigma(M^{t+1}) \end{pmatrix}$$

where $\sigma(M^i) > 0$.

Definition 5.1. *Given a filtration $0 \subset M_1 \subset \dots \subset M_{t+1} = M$ of subrepresentations of M , define $v = (v_1, \dots, v_{t+1})$, where $v_i = \Theta(M) - \frac{\sigma(M)}{\sigma(M^i)}\Theta(M^i)$, the vector associated to the filtration.*

Now we can identify the Kempf function (3.1) with the function in Theorem 4.1,

$$\begin{aligned} K(x_M, \Gamma) &= \frac{\sum_{i=1}^{t+1} \Gamma_i \cdot [\Theta(M)\sigma(M^i) - \sigma(M)\Theta(M^i)]}{\sqrt{\sum_{i=1}^{t+1} \sigma(M^i) \cdot \Gamma_i^2}} = \\ &= \frac{\sum_{i=1}^{t+1} \sigma(M^i) \Gamma_i \cdot [\Theta(M) - \frac{\sigma(M)}{\sigma(M^i)} \Theta(M^i)]}{\sqrt{\sum_{i=1}^{t+1} \sigma(M^i) \cdot \Gamma_i^2}} = \frac{(\Gamma, v)}{\|\Gamma\|} = \mu_v(\Gamma). \end{aligned}$$

Note that $\sum_{i=1}^{t+1} b^i v_i = 0$.

Lemma 5.2. [GSZ, Lemma 3.4, Lemma 3.5] *Let $0 \subset M_1 \subset \dots \subset M_{t+1} = M$ be the Kempf filtration of M (cf. Theorem 3.2). Let $v = (v_1, \dots, v_{t+1})$ the vector associated to this filtration by Definition 5.1. Then,*

- (1) *The coordinates of v verify $v_1 < v_2 < \dots < v_t < v_{t+1}$ i.e., the graph of v is convex.*
- (2) *The vector v is the convex envelope of every refinement.*

Theorem 5.3. *The Kempf filtration of M is the Harder-Narasimhan filtration of M .*

Proof. The vector v associated to the Kempf filtration of M verifies properties (1) and (2) in Lemma 5.2, which are precisely the properties (1) and (2) in Theorem 1.5, respectively. By uniqueness of the Harder-Narasimhan filtration of M , both filtrations do coincide. ■

REFERENCES

- [AC] L. Álvarez-Cónsul, *Some results on the moduli spaces of quiver bundles*, Geom. Dedicata **139** (2009), 99-120.
- [ACGP] L. Álvarez-Cónsul, O. García-Prada, *Hitchin-Kobayashi correspondence, quivers, and vortices*, Comm. Math. Phys. **238** (2003), 1-33.
- [GSZ] T. Gómez, I. Sols, A. Zamora, *A GIT characterization of the Harder-Narasimhan filtration*, arXiv:1112.1886v2, (Preprint 2011)
- [Ke] G. Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** no. 1 (1978), 299-316.
- [Ki] A. King, *Moduli of representations of finite dimensional algebras*, Quart. J. Math. Oxford Ser. (2), **45** (1994), 180, 515-530.
- [Re] M. Reineke, *Moduli of representations of quivers*, arXiv:0802.2147v1, (Preprint 2008)
- [Ru] A. Rudakov, *Stability for an abelian category*, J. Algebra, **197** (1997), 231-245.
- [Sch] A.H.W. Schmitt, *Moduli for decorated tuples of sheaves and representation spaces for quivers*, Proc. Indian Acad. Sci. Math. Sci. **115** (2005) 15-49.

INSTITUTO DE CIENCIAS MATEMÁTICAS (CSIC-UAM-UC3M-UCM), NICOLÁS CABRERA 13-15, CAMPUS CANTOBLANCO UAM, 28049 MADRID, SPAIN

DEPARTAMENTO DE ÁLGEBRA, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD COMPLUTENSE DE MADRID, 28040 MADRID, SPAIN

E-mail address: `alfonsozamora@icmat.es`